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QUADRUPOLE ROLL ERRORS AND HORIZONTAL-VERTICAL COUPLING IN THE MAIN RING

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I. INTRODUCTION

There are many indications that the horizontal and vertical coupling of the betatron oscillation in the main ring is one of the major causes of the beam loss. The effect should not be harmful if (1) the horizontal excursion of the beam is small (a good horizontal closed orbit and a good injection), and (2) resonances of the type $\nu_x + \nu_y$ = integer is avoided. The second condition is usually satisfied in the main ring even with a ripple in the quadrupole current ($\nu_x + \nu_y = 40.3 - 40.7$) but not the first one.

The largest contribution to the coupling undoubtedly arises from the roll error of the quadrupole field axis.[1] Existing data show the rms value of this roll to be approximately 5 mrad (ten times the design value given in the "white book" as the "easily attainable tolerance"), some quadrupoles having roll angles as large as 20 mrad.[2] In order to eliminate the coupling, mostly at the injection, twelve skew quadrupoles have been placed in the main ring, two in each sector at stations 27 and 42 with the maximum strength $B'\ell = \pm 120G$ per magnet. Results of preliminary trials of these skew quadrupoles have been inconclusive in their usefulness.[3]

The purpose of this report is to give an explicit expression for the relevant quantities that are responsible for the coupling when ν_x and ν_y are specified. In particular it will be pointed out that, if the coupling is strong, a compensation of one particular harmonic component alone of the roll angle is not very effective in suppressing the effect. In general, one must eliminate four parameters (equivalent of four elements of the four-dimensional transfer matrix).[1] However, one of them is usually small so that three "knobs" may be sufficient in controlling the coupling.

II. DERIVATION OF THE RELEVANT QUANTITIES

Consider an ideal ring without any imperfections except for quadrupole rolls. A number of skew quadrupoles are assumed to be in the ring. Equations for the transverse motion are

$$x'' + k_x x = f y, \qquad (1)$$

$$y'' + k_y y = f x, \qquad (2)$$

where

$k_x = k_y = f = 0$ in free space,

$k_x = 1/(\text{radius of curvature})^2$, $k_y = f = 0$ in dipoles,

$k_x = -k_y = B_Q'/(B\rho)$, $(B_Q' = \partial B_x/\partial y = \partial B_y/\partial x)$,

$f = 2\, k_x \theta$ (roll angle) in quadrupoles,

$$k_x = k_y = 0,\ f = B'_{sk}/(B\rho), (B'_{sk} = \partial B_x/\partial x = -\partial B_y/\partial y)$$
in skew quadrupoles.

For f = 0 everywhere, solutions are given by

$$x(s) = \sqrt{\beta_x(s) W_{xo}}\ \sin\ [\nu_x \phi_x(s) + \delta_x], \tag{3}$$

$$y(s) = \sqrt{\beta_y(s) W_{yo}}\ \sin\ [\nu_y \phi_y(s) + \delta_y] \tag{4}$$

Notations used here are those in "Courant-Snyder." At quadrupole positions in the main ring, $|\phi_x - \phi_y| \lesssim 1°$. In order to study the coupling effect, it is convenient to rewrite the differential equations (1) and (2) in the form of integral equations. This can be done by using four independent solutions of (1) and (2) with f = 0;

$$x_1(s) = \sqrt{\beta_x(s)}\ \sin\ [\nu_x \phi_x(s)] \tag{5}$$

$$x_2(s) = \sqrt{\beta_x(s)}\ \cos\ [\nu_x \phi_x(s)] \tag{6}$$

and $y_1(s)$ and $y_2(s)$ with analogous forms. Note that

$$x_1'\ x_2 - x_1\ x_2' = 1, \tag{7}$$

$$\alpha_x\ x_1 + \beta_x\ x_1' = x_2, \tag{8}$$

$$\alpha_x\ x_2 + \beta_x\ x_2' = -x_1. \tag{9}$$

Similar relations for y_1 and y_2 are obvious. Formal solutions of (1) and (2), which are equivalent integral equations for x(s) and y(s), can be written as

$$x(s) = x_1(s)\ A_x(s) + x_2(s)\ B_x(s), \tag{10}$$

$$y(s) = y_1(s)\ A_y(s) + y_2(s)\ B_y(s) \tag{11}$$

where

$$A_x(s) = \sqrt{W_{xo}} \cos (\delta_x) + I_{Ax}(s), \tag{12}$$

$$B_x(s) = \sqrt{W_{xo}} \sin (\delta_x) - I_{Bx}(s), \tag{13}$$

$$I_{Ax}(s) = \int_o^s ds' f(s') x_2(s') y(s'), \tag{14}$$

$$I_{Bx}(s) = \int_o^s ds' f(s') x_1(s') y(s'). \tag{15}$$

Again expressions for A_y, B_y, I_{Ay} and I_{By} are obvious. From (8) and (9),

$$\alpha_x(s)x(s) + \beta_x(s)x'(s) = x_2(s)A_x(s)-x_1(s)B_x(s), \tag{16}$$

$$\alpha_y(s)y(s) + \beta_y(s)y'(s) = y_2(s)A_y(s)-y_1(s)B_y(s). \tag{17}$$

The effect of the coupling is shown in the change in W_x and in W_y,

$$W_x(s) = \left[x^2 + (\alpha_x\ x + \beta_x\ x')^2\right] \Big/ \beta_x, \tag{18}$$

$$W_y(s) = \left[y^2 + (\alpha_y\ y + \beta_y\ y')^2\right] \Big/ \beta_y. \tag{19}$$

In the absence of the coupling, they are invariant;

$$W_x(s) = W_{xo}, \qquad W_y(s) = W_{yo}.$$

It is easy to see that

$$W_x(s) = [A_x(s)]^2 + [B_x(s)]^2, \tag{20}$$

$$W_y(s) = [A_y(s)]^2 + [B_y(s)]^2. \tag{21}$$

So far all expressions are exact; they are all equivalent to the original equations of motion (1) and (2) with specific initial conditions for x, x', y and y'. One can solve the integral equations (10) and (11), in principle, by successive iterations to any desired order of the (small) coupling parameter $f(s)$. However, expressions for $x(s)$ and $y(s)$ contain many multiple integrals and they are not useful for practical purposes.

In order to see the effect of the coupling in one turn, one substitutes the uncoupled form of $x(s)$, Eq. (3), in I_{Ay} and I_{By} and evaluates the change in W_y, Eq. (21), to the first order in $f(s)$. For $W_{yo} = 0$ ($y = y' = 0$ at $s = 0$), the lowest order is the second order in $f(s)$. If the coupling is so strong that $x(s)$ deviates from the uncoupled solution substantially in one turn, one may have to evaluate the effect separately in each sector. Conversely, for a weak coupling, one may extend this approximation to many turns. A straightforward substitution yields the following expression for $\Delta W_y \equiv W_y - W_{yo}$:

$$\Delta W_y = \sqrt{W_{xo}W_{yo}}\ \varepsilon(\delta_x)\ \sin\left[a(\delta_x) + \delta_y\right], \tag{22}$$

$$\left[\varepsilon(\delta_x)\right]^2 = S_+^2 + S_-^2 + C_+^2 + C_-^2 + 2\cos(2\delta_x)(S_+S_- - C_+C_-) + 2\sin(2\delta_x)(S_+C_- + S_-C_+), \tag{23}$$

$$a(\delta_x) = \tan^{-1}\left[\frac{S_+ + S_- + \tan(\delta_x)(C_+ + C_-)}{C_+ - C_- - \tan(\delta_x)(S_+ - S_-)}\right], \tag{24}$$

$$C_\pm = \sum_n \sqrt{\beta_{xn}\beta_{yn}}\,\ell_n f_n \cos(\nu_x\phi_{xn} \pm \nu_y\phi_{yn}), \tag{25}$$

$$S_\pm = \sum_n \sqrt{\beta_{xn}\beta_{yn}}\ \ell_n f_n \sin(\nu_x\phi_{xn} \pm \nu_y\phi_{yn}), \tag{26}$$

ℓ_n = length of main ring quadrupoles or of skew quadrupoles,

f_n = $2(B_Q'/B\rho)\ \theta$(roll angle) or $(B_{sk}'/B\rho)$ at n-th element,

ϕ_{xn}, ϕ_{yn} = horizontal and vertical phase at n-th element.

If one considers a collection of particles with the same values of W_{xo}, W_{yo} and δ_x but different values of δ_y $(0 \le \delta_y \le 2\pi)$, these particles will be initially on the boundary of an ellipse in y - y' space

$$\left(y/\sqrt{\beta_y W_{yo}}\right)^2 + \left[(\alpha_y y + \beta_y y')/\sqrt{\beta_y W_{yo}}\right]^2 = 1. \tag{27}$$

After one turn, this ellipse is deformed to

$$y/\sqrt{\beta_y W_{yo}} = R \sin\Theta(\delta_y),$$

$$(\alpha_y y + \beta_y y')/\sqrt{\beta_y W_{yo}} = R \cos\Theta(\delta_y) \tag{28}$$

where

$$0 \le \Theta \le 2\pi$$

and

$$R^2 = 1 + \sqrt{W_{xo}/W_{yo}}\ \varepsilon(\delta_x) \sin\left[a(\delta_x) + \delta_y\right]. \tag{29}$$

If all possible values of δ_x are taken as well (W_{xo} and W_{yo} still fixed), the maximum increase in W_y is given by

$$(\Delta W_y)_{max} = \sqrt{W_{xo}W_{yo}}\,\left[\varepsilon(\delta_x)\right]_{max}$$

$$= \sqrt{W_{xo}W_{yo}}\,\left(\sqrt{C_+^2 + S_+^2} + \sqrt{C_-^2 + S_-^2}\right) . \qquad (30)$$

For $W_{yo} = 0$, the maximum increase is

$$(\Delta W_y)_{max} = (W_{xo}/4)\left(\sqrt{C_+^2 + S_+^2} + \sqrt{C_-^2 + S_-^2}\right)^2 . \qquad (31)$$

Thus the amount of the coupling effect is determined by four quantities, C_+, C_-, S_+ and S_-. In principle, one must choose strengths of skew quadrupoles such that all four quantities vanish in the ring or, if the coupling is strong, within each sector separately. This is equivalent to the requirement that four elements of the two-by-two transfer matrix in (x,x', y,y') space be zero.[1]

III. DISCUSSIONS

One can rewrite (25) and (26) in a thin-lens approximation (which is more than adequate for the present purpose):

$$C_\pm = \frac{\sqrt{\beta_1\beta_2}}{(B\rho)}\left[2\ell_Q|B_Q'|\sum_n \theta_n \cos(\nu_x\pm\nu_y)\phi_n + \ell_{sk}\sum_m (B_{sk}')_m \cos(\nu_x\pm\nu_y)\phi_m\right],$$

$S_\pm$ = same as $C_\pm$ with $\sin(\nu_x\pm\nu_y)\phi_{n(m)}$ in place of cos. (32)

Here, $\sum_n$ is for main ring quadrupoles and $\sum_m$ for skew quadrupoles. Other parameters are

$$\beta_1 \simeq 100\ \mathrm{m}, \qquad \beta_2 \simeq 27\ \mathrm{m}, \qquad \ell_Q \simeq 2\ \mathrm{m},$$

$$\phi_n \simeq \phi_{xn} (\simeq \phi_{yn}).$$

The sign convention of the roll angle θ_n for a defocusing quadrupole is opposite to that for a focusing quadrupole. Expected magnitudes of the coupling are (uncorrected, $B'_{sk} = 0$)

(1) $\nu_x = 20.30, \quad \nu_y = 20.22$

$\theta_{rms} = 5$ mrad, $|\theta_n| \leq 15$ mrad,

$W_{xo} = 2$ mm-mrad (maximum beam size ±14 mm),

$W_{yo} = 0.5$ mm-mrad (maximum beam size ±7 mm)

$(\Delta W_y)_{max} = 0.47$ mm-mrad

Maximum beam size increases from ±7 mm to ~±10 mm in the vertical direction.

(2) ν_x, ν_y and $\{\theta_n\}$ as in (1),

Amplitude of the horizontal coherent motion = ±1",

no vertical motion at s = 0.

$(\Delta W_y)_{max} = 0.35$ mm-mrad which gives the maximum vertical amplitude ~±6 mm.

Depending on the actual distribution of $\{\theta_n\}$ in the ring, coupling effects could be much larger (or smaller) than indicated above.

For $\nu_x \simeq \nu_y$, S_- is very small and the cancellation of C_- amounts to the cancellation of the total roll angle $\sum_n \theta_n$. At the injection (7.2 GeV), $|B_Q'| = 5$ kG/m and twelve skew quadrupoles with $\ell_{sk}B_{sk}' = \pm 120$ G each are just what one needs to cancel C_- when the rms value of the roll angle is 5 mrad. On the other hand, the expected magnitudes of C_+ and S_+ are approximately 70% (that is, $1/\sqrt{2}$) of C_- so that, from (30) with $S_- \simeq 0$, one can hope for only 50% reduction in $(\Delta W_y)_{max}$. For $\nu_x = 20.3$ and $\nu_y = 20.22$, a typical operating point of the main ring,

$$|C_+| \simeq |C_-| \simeq |S_+|, \qquad |S_-/C_-| \simeq 1/3.$$

Under these conditions, $(\Delta W_y)_{max}$ is reduced by only ~30% if C_- alone is cancelled. When $C_\pm$ and S_+ are eliminated, the reduction in $(\Delta W_y)_{max}$ is expected to be almost 90%. It seems desirable to have at least three groups of skew quadrupoles, each group having the total strength $\ell_{sk} \sum B_{sk}' \simeq \pm 1.5$ kG or preferably more. Skew quadrupoles distributed around the ring next to all quadrupoles are, of course, the ideal solution.

REFERENCES

1. L. C. Teng, TM-382, July 21, 1972.

2. A. G. Ruggiero, private communications.

3. S. Mori and R. Stiening, private communications.